

## NONCOMMUTATIVE COMPLETE INTERSECTIONS

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ABSTRACT. Several generalizations of a commutative ring that is a graded complete intersection are proposed for a noncommutative graded  $k$ -algebra; these notions are justified by examples from noncommutative invariant theory.

## 0. INTRODUCTION

Bass has noted that Gorenstein rings are ubiquitous [B]. Since the class of Gorenstein rings contains a wide variety of rings, it has proven useful to consider a tractable class of Gorenstein rings, and complete intersections fill that role for commutative rings. Similarly Artin-Schelter Gorenstein algebras, which are noncommutative generalizations of commutative Gorenstein rings, include a diverse collection of algebras, and finding a class of Artin-Schelter Gorenstein algebras that generalizes the class of commutative complete intersections is an open problem in noncommutative algebra. In this paper we use our work in noncommutative invariant theory to propose several notions of a noncommutative graded complete intersection. Moreover, the existence of noncommutative analogues of commutative complete intersection invariant subalgebras broadens our continuing project of establishing an invariant theory for finite groups acting on Artin-Schelter regular algebras that is parallel to classical invariant theory (see [KKZ1]–[KKZ5]).

When a finite group acts linearly on a commutative polynomial ring, the invariant subring is rarely a regular ring (the group must be a reflection group [ShT]), but Gorenstein rings of invariants are easily produced. For example, Watanabe's Theorem ([W1] or [Be, Theorem 4.6.2]) states that the invariant subring of  $k[x_1, \dots, x_n]$  under the natural action of a finite subgroup of  $SL_n(k)$  is always Gorenstein, where  $k$  is a base field. In previous work we have shown there is a rich invariant theory for finite group (and even Hopf) actions on Artin-Schelter regular [Definition 3.1] (or AS regular, for short) algebras; for example there is a noncommutative version of Watanabe's Theorem [JoZ, Theorem 3.3], providing conditions when the invariant subring is AS Gorenstein.

Cassidy and Vancliff defined a factor ring  $S/I$  of  $S = k_{(q_{ij})}[x_1, \dots, x_n]$ , a skew polynomial ring, to be a complete intersection if  $I$  is generated by a regular sequence of length  $n$  in  $S$  (hence  $S/I$  is a finite dimensional algebra) [CV, Definition 3.7], and in [V] Vancliff considered extending this definition to graded skew Clifford algebras. A few examples of noncommutative (or quantum) complete intersections have been constructed and studied along the line of factoring out a regular sequence of elements [BE, BO, Op]. Further, different kinds of generalizations of a

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commutative complete intersection have been proposed during the last fifteen years [Go, EG, BG, GHS].

In the commutative graded case a connected graded algebra  $A$  is called a *complete intersection* if one of the following four equivalent conditions holds [Lemma 1.7]

- (cci')  $A \cong k[x_1, \dots, x_d]/(\Omega_1, \dots, \Omega_n)$ , where  $\{\Omega_1, \dots, \Omega_n\}$  is a regular sequence of homogeneous elements in  $k[x_1, \dots, x_d]$  with  $\deg x_i > 0$ .
- (cci)  $A \cong C/(\Omega_1, \dots, \Omega_n)$ , where  $C$  is a noetherian AS regular algebra and  $\{\Omega_1, \dots, \Omega_n\}$  is a regular sequence of normalizing homogeneous elements in  $C$ .
- (gci) The Ext-algebra  $E(A) := \bigoplus_{n=0}^{\infty} \text{Ext}_A^n(k, k)$  of  $A$  has finite Gelfand-Kirillov dimension.
- (nci) The Ext-algebra  $E(A)$  is noetherian.

In the noncommutative case, unfortunately, these four conditions are not all equivalent, nor do (gci) or (nci) force  $A$  to be Gorenstein, making it unclear which property to use as the proper generalization of a commutative complete intersection. A direct generalization to the noncommutative case is condition (cci) which involves considering regular sequences in *any* AS regular algebra (in the commutative case the only AS regular algebras are the polynomial algebras). Several researchers, including those whom we have mentioned earlier, have taken an approach that uses regular sequences. Though the condition (cci) seems to be a good definition of a noncommutative complete intersection, there are very few tools available to work with condition (cci) except for explicit construction and computation, and it is not easy to show condition (cci) fails since one needs to consider regular sequences in *any* AS regular algebra.

We consider both conditions (gci) and (nci) as possible definitions of a noncommutative complete intersection. One advantage of this approach is that it covers a large class of examples coming from noncommutative graded ring theory. For example, let  $R$  be any noetherian Koszul algebra of finite Gelfand-Kirillov dimension (or GK-dimension, for short), then the Koszul dual  $A := R^!$  satisfies both (gci) and (nci), since the Ext-algebra  $E(A)$  of  $A$  is  $A^! = R^{!!} = R$ , which is noetherian of finite GK-dimension. We provide some information about the relationship between these different notions of noncommutative complete intersection in the following theorem.

**Theorem 0.1.** [Theorem 1.11(a).] *Let  $A$  be a connected graded noncommutative algebra. If  $A$  satisfies (cci), then it satisfies (gci).*

Other connections between the conditions (cci), (gci) and (nci) are studied in the paper.

Many interesting examples arise from the noncommutative invariant theory of AS regular algebras under finite group actions, and the following question was one motivation for our work. Let  $\text{Aut}(A)$  denote the group of graded algebra automorphisms of  $A$ .

**Question 0.2.** Let  $A$  be a (noncommutative) noetherian connected graded AS regular algebra and  $G$  be a finite subgroup of  $\text{Aut}(A)$ . Under what conditions is the invariant subring  $A^G$  a complete intersection?

When  $A$  is a commutative polynomial ring over  $\mathbb{C}$ , Question 0.2 was answered by Gordeev [G2] and Nakajima [N3, N4] independently. A very important tool in

the classification of groups  $G$ , such that  $k[x_1, \dots, x_n]^G$  is a complete intersection, is a result of Kac and Watanabe [KW] and Gordeev [G1] independently; they prove that if  $G$  is a finite subgroup of  $GL_n(k)$  and  $k[x_1, \dots, x_n]^G$  is a complete intersection, then  $G$  is generated by bireflections (i.e. elements  $g \in G$  such that  $\text{rank}(g - I) \leq 2$ ). This leads us to the next natural question: what is a bireflection in the noncommutative setting? We seek notions of complete intersection and bireflection that lead to a generalization of the result of Kac-Watanabe-Gordeev.

Our generalization of a reflection of a symmetric algebra to the notion of a quasi-reflection of a noncommutative AS regular algebra [KKZ1, Definition 2.2], suggests that a reasonable definition of a noncommutative bireflection is following: a graded algebra automorphism  $g$  of a noetherian AS regular algebra  $A$  of GK-dimension  $n$  is called a *quasi-bireflection* if its trace has the form:

$$\text{Tr}_A(g, t) := \sum_{i=0}^{\infty} \text{Tr}(g|_{A_i}) t^i = \frac{1}{(1-t)^{n-2} q(t)}$$

where  $q(1) \neq 0$ . As in the case of quasi-reflections, there are “mystic quasi-bireflections” (quasi-bireflections that are not bireflections of  $A_1$ ) [Example 6.6]. We prove the following theorem in Section 4 by reducing to the commutative case.

**Theorem 0.3.** *Let  $A$  be the quantum affine space  $k_q[x_1, \dots, x_d]$  such that  $q \neq \pm 1$ . Let  $G$  be a finite subgroup of  $\text{Aut}(A)$ . If  $A^G$  satisfies (gci), then  $G$  is generated by quasi-bireflections.*

Some partial results about  $k_{-1}[x_1, \dots, x_d]^G$  are given in [KKZ5]. Further evidence that the definition of quasi-bireflection is a useful generalization is illustrated by Theorems 0.4 and 0.5 below.

Noetherian graded down-up algebras form a class of AS regular algebras of global dimension 3. Let  $Q_3$  be the finite subgroup  $GL_2(k)$  generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where  $\epsilon$  is a primitive  $n$ th root of unity for an odd integer  $n \geq 3$ . Let  $Q_4$  be the finite subgroup  $GL_2(k)$  generated by

$$\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where  $\epsilon$  is a primitive  $4n$ th root of unity.

An algebra  $A$  is called *cyclotomic Gorenstein* if (i)  $A$  is AS Gorenstein and (ii) the Hilbert series  $H_A(t)$  of  $A$  is a rational function of the form  $p(t)/q(t)$  and the roots of  $p(t)$  and  $q(t)$  are roots of unity. A commutative complete intersection is cyclotomic Gorenstein.

**Theorem 0.4.** [KKZ4, Theorem 0.2] *Assume that  $k$  is of characteristic zero. Let  $A$  be a connected graded noetherian down-up algebra and  $G$  be a finite subgroup of  $\text{Aut}(A)$ .*

- (a) *If  $A^G$  satisfies (gci), then  $G$  is generated by quasi-bireflections.*
- (b) *The following conditions are equivalent:*
  - (i)  *$A^G$  satisfies (gci).*
  - (ii)  *$A^G$  is cyclotomic Gorenstein.*

- (iii)  $G$  is a finite subgroup of  $GL_2(k)$  such that (iiia)  $\det g = 1$  or  $-1$  for each  $g \in G$  and (iiib)  $G$  is not conjugate to  $Q_3$  and  $Q_4$  defined as above.

Theorem 0.4 suggests that the “cyclotomic Gorenstein” property is closely related to our notions of complete intersection for noncommutative algebras of the form  $A^G$  where  $A$  is a noetherian AS regular algebra and  $G \subset \text{Aut}(A)$ . For the generic 3-dimensional Sklyanin algebra  $A := A(a, b, c)$ , we also show that  $A^G$  is cyclotomic Gorenstein if and only if  $G$  is generated by quasi-bireflections [Theorem 5.5]. Here is another result of this kind.

**Theorem 0.5.** [KKZ5, Theorem 4.8] *Assume that  $k$  is of characteristic zero. Let  $A = k_{-1}[x_1, \dots, x_n]$  and  $G$  be a subgroup of the permutation group  $S_n$  (acting naturally on  $A$ ). If  $G$  is generated by quasi-bireflections, then  $A^G$  satisfies (cci).*

In the commutative case, we have [St3, p. 506]

$$\begin{aligned} \text{regular} &\implies \text{hypersurface} \implies \text{complete intersection} \\ &\implies \text{Gorenstein} \implies \text{Cohen-Macaulay.} \end{aligned}$$

In contrast to the commutative case, neither (gci) nor (nci) implies Gorenstein in the noncommutative case [Example 6.3]. On the other hand, when we work with invariant subrings  $A^G$ , we have a satisfactory situation.

**Theorem 0.6.** [Theorem 3.4.] *Assume that  $k$  is of characteristic zero. Let  $A$  be a connected graded noetherian Auslander regular algebra and let  $G$  be a finite subgroup of  $\text{Aut}(A)$ . If  $A^G$  satisfies any one of (cci), or (gci), or (nci), then  $A^G$  is cyclotomic Gorenstein.*

It follows that when  $A^G$  is not cyclotomic Gorenstein we know that it does not satisfy any of our conditions for generalizations of complete intersections. In Section 2 we use the cyclotomic Gorenstein condition to show that certain Veronese subrings are AS Gorenstein algebras that are not any of our kinds of a complete intersection.

## 1. NONCOMMUTATIVE VERSIONS OF COMPLETE INTERSECTION

Throughout let  $k$  be a base commutative field of characteristic zero. Vector spaces, algebras and morphisms are over  $k$ .

In this section we propose several different, but closely related, definitions of a noncommutative complete intersection graded  $k$ -algebra. We begin by recalling some definitions that will be used in our work. The Hilbert series of an  $\mathbb{N}$ -graded  $A$ -module  $M = \bigoplus_{i=0}^{\infty} M_i$  is defined to be the formal power series in  $t$

$$H_M(t) = \sum_{i=0}^{\infty} (\dim M_i) t^i.$$

**Definition 1.1.** Let  $A = \bigoplus_{i=0}^{\infty} A_i$  be a connected graded algebra.

- (a)  $A$  has *exponential growth* if  $\overline{\lim}_i (\dim A_i)^{1/i} > 1$ .
- (b)  $A$  has *sub-exponential growth* if  $\overline{\lim}_i (\dim A_i)^{1/i} \leq 1$ .
- (c) The *Gelfand-Kirillov dimension* (or *GK-dimension*, for short) of  $A$  is

$$\text{GKdim } A = \overline{\lim}_n \log_n (\dim \bigoplus_{i=0}^n A_i).$$

We refer to [KL] for the original definition of GK-dimension and its basic properties. Definition 1.1(c) agrees with the original definition given in [KL] when  $A$  is finitely generated, but it may differ otherwise.

Next are the proposed definitions of a noncommutative complete intersection. The definition of AS regularity will be reviewed in Section 3. By “an element in a graded algebra” we will mean a homogeneous element. A set of (homogeneous) elements  $\{\Omega_1, \dots, \Omega_n\}$  in a graded algebra  $A$  is called a *sequence of regular normal elements* if, for each  $i$ , the image of  $\Omega_i$  in  $A/(\Omega_1, \dots, \Omega_{i-1})$  is regular (i.e. a non-zero-divisor in  $A/(\Omega_1, \dots, \Omega_{i-1})$ ) and normal in  $A/(\Omega_1, \dots, \Omega_{i-1})$  (i.e. the left ideal generated by its image is a two-sided ideal in  $A/(\Omega_1, \dots, \Omega_{i-1})$ ).

**Definition 1.2.** Let  $A$  be a connected graded finitely generated algebra.

- (a) We say  $A$  is a *classical complete intersection* (or *cci*, for short) if there is a connected graded noetherian AS regular algebra  $C$  and a sequence of regular normal elements  $\{\Omega_1, \dots, \Omega_n\}$  of positive degree such that  $A \cong C/(\Omega_1, \dots, \Omega_n)$ .
- (b) The *cci number* of  $A$  is defined to be

$$cci(A) = \min\{n \mid A \cong C/(\Omega_1, \dots, \Omega_n) \text{ as in part (a)}\}.$$

- (c) We say  $A$  is a *hypersurface* if  $cci(A) \leq 1$ .

Let  $E(A)$  denote the Ext-algebra  $\text{Ext}_A^*(k, k)$  of  $A$ . It is  $\mathbb{Z}^2$ -graded and can be viewed as a connected graded algebra by using either the cohomological degree or the Adams grading from  $A$ .

**Definition 1.3.** Let  $A$  be a connected graded finitely generated algebra.

- (a) We say  $A$  is a *complete intersection of noetherian type* (or *nci*, for short) if the Ext-algebra  $E(A)$  is a left and right noetherian ring.
- (b) When  $A$  is a nci, the *nci number* of  $A$  is defined to be

$$nci(A) = \text{Kdim } E(A)_{E(A)}$$

where  $\text{Kdim}$  is the Krull dimension (of a right module).

- (c) We say  $A$  is an *n-hypersurface* if  $nci(A) \leq 1$ .

In the next definition, we consider  $E(A) = \bigoplus_{i=0}^{\infty} \text{Ext}_A^i(k, k)$  as a connected graded algebra by using the cohomological degree, so that the degree  $i$  piece is  $E_i = \text{Ext}_A^i(k, k)$ .

**Definition 1.4.** Let  $A$  be a connected graded finitely generated algebra.

- (a) We say  $A$  is a *complete intersection of growth type* (or *gci*, for short) if  $E(A)$  has finite GK-dimension.
- (b) The *gci number* of  $A$  is defined to be

$$gci(A) = \text{GKdim}(E(A)).$$

- (c) We say  $A$  is a *g-hypersurface* if  $gci(A) \leq 1$ .
- (d) We say  $A$  is a *weak complete intersection of growth type* (or *wci*) if  $E(A)$  has subexponential growth.

If  $E(A)$  has finite GK-dimension then it has subexponential growth. Hence a gci must be a wci. We recall the  $\chi$ -condition. Let  $k$  also denote the graded  $A$ -bimodule  $A/A_{\geq 1}$ .

**Definition 1.5.** [AZ, Definition 3.2] Let  $A$  be a noetherian, connected graded algebra. We say that the  $\chi$ -condition holds for  $A$  if  $\text{Ext}_A^j(k, M)$  is finite dimensional over  $k$  for all finitely generated graded left (and right)  $A$ -modules  $M$  and all  $j \geq 0$ .

The following lemma is easy.

**Lemma 1.6.** *Let  $A$  be connected graded. Then the following are equivalent.*

- (a)  *$A$  has finite global dimension and every term in the minimal free resolution of the trivial module  $k$  is finitely generated.*
- (b)  *$\text{Ext}_A^i(k, k)$  is finite dimensional for all  $i$  and  $\text{Ext}_A^i(k, k) = 0$  for all  $i \gg 0$ .*
- (c)  *$\text{nci}(A) = 0$ .*
- (d)  *$\text{gci}(A) = 0$ .*

*If, further,  $A$  is noetherian and satisfies the  $\chi$ -condition, then the following are also equivalent to the above.*

- (e)  *$A$  is AS regular.*
- (f)  *$\text{cci}(A) = 0$ .*

*Proof.* (a)  $\iff$  (e): This follows from [Zh1, Theorem 0.3]. The rest is straightforward.  $\square$

In the commutative case, the different versions of complete intersection are equivalent.

**Lemma 1.7.** *Let  $A$  be a connected graded finitely generated commutative algebra. Then the following are equivalent.*

- (a)  *$A \cong k[x_1, \dots, x_d]/(\Omega_1, \dots, \Omega_n)$ , where  $\{\Omega_1, \dots, \Omega_n\}$  is a sequence of regular homogeneous elements in  $k[x_1, \dots, x_d]$  with  $\deg x_i > 0$ .*
- (b)  *$A$  is a cci.*
- (c)  *$A$  is a nci.*
- (d)  *$A$  is a gci.*

*Proof.* By [BH, FHT, FT, Gu, T], it is well-known that (a), (c) and (d) are equivalent for local commutative rings. The current graded version follows from [FT, Theorem IV.7] and [FHT, Theorem C].

(a)  $\implies$  (b): This is obvious.

(b)  $\implies$  (d): This follows from Theorem 1.11 which is to be proved later.  $\square$

In the noncommutative case, 1.7(a) and 1.7(b,c,d) are clearly not equivalent. Example 6.2 gives an algebra that is a gci, but is not a nci or a cci. Example 6.3 gives an algebra that is both a gci and a nci, but is not a cci.

Every commutative complete intersection ring  $A := k[x_1, \dots, x_d]/(\Omega_1, \dots, \Omega_n)$  is Gorenstein and its Hilbert series is of the form

$$H_A(t) = \frac{\prod_{j=1}^n (1 - t^{\deg \Omega_j})}{\prod_{i=1}^d (1 - t^{\deg x_i})}.$$

We use a similar condition to define a related notion.

**Definition 1.8.** Let  $A$  be a connected graded finitely generated algebra.

- (a) We say a rational function  $p(t)/q(t)$ , where  $p(t)$  and  $q(t)$  are coprime integral polynomials, is *cyclotomic* if all the roots of  $p(t)$  and  $q(t)$  are roots of unity.
- (b) We say  $A$  is *cyclotomic* if its Hilbert series  $H_A(t)$  is of the form  $p(t)/q(t)$  and it is cyclotomic.

(c) The *cyc* number of  $A$  is defined to be

$$\text{cyc}(A) = \min\{m \mid H_A(t) = \frac{\prod_{i=1}^m (1 - t^{a_i})}{\prod_{j=1}^n (1 - t^{b_j})}\}.$$

(d) We say  $A$  is *cyclotomic Gorenstein* if the following conditions hold

- (i)  $A$  is AS Gorenstein;
- (ii)  $A$  is cyclotomic.

Let  $A$  be the commutative ring  $k[x, y, z]/(xy)$  where  $\deg x = \deg y = 1$  and  $\deg z = 2$ . Then  $\text{cyc}(A) = 0$ , but  $\text{cci}(A) = \text{nci}(A) = \text{gci}(A) = 1$ . It also is easy to construct algebras  $A$  such that  $\text{cyc}(A) = 0$  and  $\text{cci}(A) = \text{nci}(A) = \text{gci}(A)$  is any positive integer. The notion of cyclotomic Gorenstein is weaker than the notion of complete intersection, even in the commutative case, as seen in Example 6.4, an example given by Stanley. Thus we have the following implications in the commutative case.

$$(E1.8.1) \quad \text{cci} \iff \text{gci} \iff \text{nci} \implies \text{cyclotomic Gorenstein}.$$

Next we begin to relate these conditions in the noncommutative case.

**Lemma 1.9.** *Let  $A$  be connected graded and finitely generated. Suppose  $E := E(A)$  is noetherian. Then  $\text{nci}(A) = 1$  if and only if  $\text{gci}(A) = 1$ .*

*Proof.* If  $\text{gci}(A) = \text{GKdim } E = 1$  (and as  $E$  is noetherian),  $E$  is PI by a result of Small-Stafford-Warfield [SSW]. Hence  $\text{Kdim } E_E = \text{GKdim } E = 1$ . Conversely, assume that  $\text{Kdim } E_E = \text{nci}(A) = 1$ . It suffices to show that  $\text{GKdim } E = 1$ . Since  $E$  is noetherian, we only need to show  $\text{GKdim } E/\mathfrak{p} \leq 1$  for every prime ideal  $\mathfrak{p}$  of  $E$ . When  $R := E/(\mathfrak{p})$  is prime there is a homogeneous regular element  $f$  in  $R$ . Since  $\text{Kdim } R \leq \text{Kdim } E = 1$ ,  $\text{Kdim } R/Rf = 0$ . Thus  $R/Rf$  is finite dimensional. So  $R$  has GK-dimension 1.  $\square$

It is unknown if  $\text{nci}(A) = 2$  is equivalent to  $\text{gci}(A) = 2$  when  $E$  is noetherian.

We say a sequence of nonnegative numbers  $\{b_n\}_{n=0}^\infty$  has subexponential growth (or the formal power series  $\sum_{n=0}^\infty b_n t^n$  has subexponential growth) if  $\lim_i (b_i)^{1/i} \leq 1$ . By [StZ, Lemma 1.1],  $\{b_n\}_{n=0}^\infty$  has subexponential growth if and only if  $\{c_n := \sum_{i=0}^n b_i\}_{n=0}^\infty$  has subexponential growth. Here is our first main theorem.

**Theorem 1.10.** *Suppose  $A$  is connected graded and finitely generated. Let  $\Omega$  be a regular normal element in  $A$  of positive degree and let  $B = A/(\Omega)$ .*

(a)  $\text{gci}(A) \leq \text{gci}(B) \leq \text{gci}(A) + 1$ , or equivalently,

$$\text{GKdim Ext}_A^*(k, k) \leq \text{GKdim Ext}_B^*(k, k) \leq \text{GKdim Ext}_A^*(k, k) + 1.$$

(b)  $A$  is a wci if and only if  $B$  is.

(c) If  $\text{cci}(B) = 1$ , then  $B$  is both a gci and a nci, and  $\text{gci}(B) = \text{nci}(B) = 1$ .

*Proof.* (a) Let  $a_n = \dim \text{Tor}_n^A(k, k)$  and  $b_n = \dim \text{Tor}_n^B(k, k)$ . Since  $\text{Tor}_n^A(k, k)^* \cong \text{Ext}_A^n(k, k)$ , we have  $a_n = \dim \text{Ext}_A^n(k, k)$  and  $b_n = \dim \text{Ext}_B^n(k, k)$ .

Since  $B$  is a factor ring of  $A$ , there is a graded version of the change-of-rings spectral sequence given in [Ro, Theorem 10.71]

$$(E1.10.1) \quad {}^2E_{pq} := \text{Tor}_p^B(k, \text{Tor}_q^A(B, k)) \implies_p \text{Tor}_n^A(k, k).$$

Since  $B = A/(\Omega)$  where  $\Omega$  is a regular element in  $A$ ,  $\text{Tor}_0^A(B, k) = k$ ,  $\text{Tor}_1^A(B, k) = k$  and  $\text{Tor}_i^A(B, k) = 0$  for  $i > 1$ . Hence the  $E^2$ -page of the spectral sequence (E1.10.1) has only two possibly non-zero rows; namely

$$\begin{aligned} q = 0 : & \quad \text{Tor}_p^B(k, k) \text{ for } p = 0, 1, 2, \dots, \text{ and} \\ q = 1 : & \quad \text{Tor}_p^B(k, k) \text{ for } p = 0, 1, 2, \dots \end{aligned}$$

Since (E1.10.1) converges, we have  $\text{Tor}_0^B(k, k) = \text{Tor}_0^A(k, k) = k$  and a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Tor}_4^B(k, k) &\longrightarrow \text{Tor}_2^B(k, k) \longrightarrow \\ \longrightarrow \text{Tor}_3^A(k, k) &\longrightarrow \text{Tor}_3^B(k, k) \longrightarrow \text{Tor}_1^B(k, k) \longrightarrow \\ \longrightarrow \text{Tor}_2^A(k, k) &\longrightarrow \text{Tor}_2^B(k, k) \longrightarrow \text{Tor}_0^B(k, k) \longrightarrow \\ \longrightarrow \text{Tor}_1^A(k, k) &\longrightarrow \text{Tor}_1^B(k, k) \longrightarrow 0. \end{aligned}$$

As a part of the above long exact sequence, we have, for each  $n$ , the following two exact sequences

$$(E1.10.2) \quad \text{Tor}_{n-1}^B(k, k) \longrightarrow \text{Tor}_n^A(k, k) \longrightarrow \text{Tor}_n^B(k, k),$$

and

$$(E1.10.3) \quad \text{Tor}_{n+2}^A(k, k) \longrightarrow \text{Tor}_{n+2}^B(k, k) \longrightarrow \text{Tor}_n^B(k, k) \longrightarrow \text{Tor}_{n+1}^A(k, k).$$

Then (E1.10.2) implies that

$$(E1.10.4) \quad a_n \leq b_n + b_{n-1}$$

for all  $n$ , and (E1.10.3) implies that

$$(E1.10.5) \quad |b_{n+2} - b_n| \leq a_{n+2} + a_n$$

for all  $n$ . Using Definition 1.1(c), (E1.10.4) implies that

$$\text{GKdim Ext}_A^*(k, k) \leq \text{GKdim Ext}_B^*(k, k),$$

and (E1.10.5) implies that

$$\text{GKdim Ext}_B^*(k, k) \leq \text{GKdim Ext}_A^*(k, k) + 1.$$

(b) If  $b_n$  has subexponential growth, then (E1.10.4) implies that  $a_n$  has subexponential growth. On the other hand, if  $a_n$  has subexponential growth, by [StZ, Lemma 1.1],  $c_n := \sum_{i=0}^n a_i$  has subexponential growth. Now (E1.10.5) implies that  $b_n$  has subexponential growth. Thus the assertion in (b) follows.

(c) We prove this case assuming only that  $A$  has finite global dimension (we do not need to use the fact that  $A$  is AS regular).

Let  $E^n(A) = \text{Ext}_A^n(k, k)$  and  $E^n(B) = \text{Ext}_B^n(k, k)$  for every  $n$ . There is a spectral sequence for Ext-groups, similar to (E1.10.1), which gives rise to a long exact sequence, see [CS, Theorem 6.3 and the discussion afterwards],

$$(E1.10.6) \quad E^{n-1}(A) \xrightarrow{\gamma} E^{n-2}(B) \xrightarrow{d_2} E^n(B) \xrightarrow{\phi^*} E^n(A).$$

When set  $n = p$ , this is the exact sequence [CS, (6.1)]. Then [CS, Lemma 6.1 and Theorem 6.3] show that (E1.10.6) can be interpreted as an exact triangle of right  $E(B)$ -modules. There is another sequence that can be interpreted as an exact triangle of left  $E(B)$ -modules, see [CS, (6.2)] and the discussion after [CS, Theorem 6.3]. For our purpose, it is enough to use that  $d_2 : E(B) \rightarrow E(B)$  is a right  $E(B)$ -module homomorphism, which is special case of [CS, Lemma 6.1]. Therefore  $d_2$  is a left multiplication  $l_x$  by some element  $x \in E^2(B)$ . Since  $A$  has finite global dimension  $E^n(A) = 0$  for all  $n \gg 0$ . By the exact sequence (E1.10.6) and the fact that  $E^n(A) = 0$  for all  $n \gg 0$ , we have that  $E(B)/xE(B)$  is finite dimensional. Thus  $E(B)$  is a finitely generated left  $k[x]$ -module. Therefore  $E(B)$



is left noetherian of GK-dimension  $\geq 1$ . By symmetry,  $E(B)$  is right noetherian, too. Therefore  $E(B)$  is left and right noetherian, and  $B$  is a nci.

By part (a)  $gci(B) \leq 1$ . By Lemma 1.6,  $gci(B) \neq 0$ . By Lemma 1.9,  $gci(A) = nci(A) = 1$ , and consequently  $A$  is both a n-hypersurface and a g-hypersurface.  $\square$

Under the hypotheses of Lemma 1.6 we have

$$(E1.10.7) \quad cci(A) = 0 \iff gci(A) = 0 \iff nci(A) = 0.$$

By Lemma 1.9 and Theorem 1.10(c), if  $A$  is a nci, then

$$(E1.10.8) \quad cci(A) = 1 \implies nci(A) = 1 \iff gci(A) = 1.$$

It is unknown if  $gci(A) = 1$  implies that  $nci(A) = 1$  (without assuming  $A$  is a nci a priori) and if  $gci(A) = 1$  implies that  $cci(A) = 1$ . It would be nice to have examples to answer these basic questions.

Here is our second main result, and part (a) is Theorem 0.1.

**Theorem 1.11.** *Let  $A$  be a connected graded finitely generated algebra.*

- (a) *If  $A$  is a cci, then  $A$  is a gci, and  $gci(A) \leq cci(A)$ .*
- (b) *If  $A$  is a gci or a nci, then  $A$  is a wci.*
- (c) *Suppose that  $A$  is a noetherian wci and that the Hilbert series  $H_A(t)$  is a rational function  $p(t)/q(t)$  for some coprime integral polynomials  $p(t), q(t) \in \mathbb{Z}[t]$  with  $p(0) = q(0) = 1$ . Then  $A$  is cyclotomic.*

*Proof.* (a) Write  $A = C/(\Omega_1, \dots, \Omega_n)$  for a noetherian AS regular algebra  $C$  and a sequence of regular normal elements  $\{\Omega_1, \dots, \Omega_n\}$  of positive degree. The assertion follows from Theorem 1.10(a) and the induction on  $n$ .

(b) Let  $E = E(A)$ . If  $A$  is a gci, then  $\text{GKdim } E < \infty$ . This implies that  $E$  has subexponential growth. Hence  $A$  is a wci.

If  $A$  is a nci, then  $E$  is noetherian. By [StZ, Theorem 0.1],  $E$  has subexponential growth. Therefore  $A$  is a wci.

(c) Consider a minimal free resolution of the trivial module

$$(E1.11.1) \quad \dots \rightarrow P^i \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow k \rightarrow 0$$

with  $P^i = \bigoplus_{s=1}^{n_i} A(-d_s^i)$  for all  $i \geq 0$ . In particular,  $P^0 = A$ . The Hilbert series of  $P^i$  is

$$H_{P^i}(t) = \sum_{s=1}^{n_i} t^{d_s^i} H_A(t) = \left( \sum_{s=1}^{n_i} t^{d_s^i} \right) H_A(t).$$

Using the additive property of the  $k$ -dimension, the exact sequence (E1.11.1) implies that

$$1 = \sum_{i=0}^{\infty} (-1)^i H_{P^i}(t) = \sum_{i=0}^{\infty} (-1)^i \left( \sum_{s=1}^{n_i} t^{d_s^i} \right) H_A(t),$$

which implies that

$$H_A(t) = \frac{1}{\sum_{i=0}^{\infty} (-1)^i \left( \sum_{s=1}^{n_i} t^{d_s^i} \right)} =: \frac{1}{Q(t)}.$$

Since  $H_A(t) = p(t)/q(t)$ ,  $Q(t) = q(t)/p(t)$ . Since (E1.11.1) is a minimal resolution of  $k$ , we obtain that  $\text{Ext}_A^i(k, k) = \bigoplus_{s=1}^{n_i} k(d_s^i)$  for all  $i$ . Clearly the Ext-algebra  $E := \text{Ext}_A^*(k, k)$  is  $\mathbb{Z} \times \mathbb{Z}$ -graded and the every nonzero element in  $k(d_s^i)$  has degree  $(-d_s^i, i)$ , where the first grading is the Adams grading coming from the grading of  $A$  and the second grading is the cohomological grading. Since

(E1.11.1) is minimal,  $d_s^i \geq i$  for all  $s$ . Reassigning degree by  $\deg(k(d_s^i)) = (d_s^i, i)$ ,  $E$  becomes connected and  $\mathbb{N} \times \mathbb{N}$ -graded. This regrading is equivalent to letting  $\text{Ext}_A^i(k, k) = \bigoplus_{s=1}^{n_i} k(-d_s^i)$ . In this case the  $\mathbb{Z} \times \mathbb{Z}$ -graded Hilbert series of  $E$  now is

$$H_E(t, u) = \sum_{i=0}^{\infty} \left( \sum_{s=1}^{n_i} t^{d_s^i} \right) u^i.$$

By definition of  $Q(t)$ , we have  $Q(t) = H_E(t, -1)$ . By hypotheses,  $A$  is a wci, so  $E$  has subexponential growth. Hence the Hilbert series  $H_E(1, u)$  has subexponential growth. Write  $H_E(1, u) = \sum_{i \geq 0} e_i u^i$ , where  $e_i = \dim \text{Ext}_A^i(k, k)$ , and let  $F_n = \sum_{i=0}^n e_i$ . By [StZ, Lemma 1.1(1)],  $\{F_n\}_n$  has subexponential growth. Write  $E(t, 1) = \sum_{n \geq 0} f_n t^n$ , where  $f_n$  is the number of  $d_s^i$  that are equal to  $n$ . Since each  $d_s^i \geq i$  for all  $s = 1, \dots, n_i$ ,  $\sum_{i=0}^n f_n \leq F_n$  for all  $n$ . Since  $\{F_n\}_n$  has subexponential growth, so does  $\{f_n\}_n$ . Since the absolute value of each coefficient of the power series  $H_E(t, -1)$  is no more than (the absolute value of) each coefficient of the power series  $H_E(t, 1)$ ,  $H_E(t, -1)$  has subexponential growth. As noted before  $Q(t) = H_E(t, -1)$ , and we conclude that the coefficients of  $Q(t)$  have subexponential growth. We have seen that  $Q(t) = q(t)/p(t)$  and write  $p(t) = \prod_{i=1}^d (1 - r_i t)$ . By [StZ, Lemma 2.1],  $|r_i| \leq 1$ . Since  $p(t)$  has integral coefficients,  $|r_i| = 1$  for all  $i$ . The proof of [StZ, Corollary 2.2] shows that each  $r_i$  is a root of unity.

Since  $H_A(t) = p(t)/q(t)$ , by [StZ, Corollary 2.2], each root of  $q(t)$  is a root of unity. Therefore  $A$  is cyclotomic.  $\square$

By Theorem 1.11 we have the implications below.

(E1.11.2)

$$\begin{array}{ccc}
 cci & & nci \\
 \text{Theorem 1.11(a)} \downarrow & & \downarrow \text{Theorem 1.11(b)} \\
 gci & \xrightarrow{\text{Theorem 1.11(b)}} & wci \\
 \text{assume } H_A(t) \text{ is rational} \downarrow \text{Theorem 1.11(c)} & & \\
 & & \text{cyclotomic}
 \end{array}$$

## 2. HIGHER VERONESE SUBRINGS ARE NOT CYCLOTOMIC

In many examples, one proves that an algebra  $A$  is not a complete intersection of any type by showing that  $A$  is not cyclotomic, see the diagram (E1.11.2). In this short section we show that most Veronese subrings of quantum polynomial rings are not cyclotomic. First we will use a very nice result of Brenti-Welker [BW, Theorem 1.1]. Recall from [BW] that for  $a, b, c \in \mathbb{N}$  the partition number is defined by

$$C(a, b, c) = |\{(n_1, \dots, n_b) \in \mathbb{N}^b \mid \sum_{i=1}^b n_i = c, 0 \leq n_i \leq a \forall i\}|.$$

For example,  $C(1, 3, 1) = 3$ .

**Lemma 2.1.** [BW, Theorem 1.1] *Let  $(a_n)_{n \geq 0}$  be a sequence of complex numbers such that for some  $s, d \geq 0$  its generating series  $f(t) := \sum_{n \geq 0} a_n t^n$  satisfies*

$$f(t) = \frac{h_0 + \cdots + h_s t^s}{(1-t)^d}.$$

*Set  $f^{(r)}(t) = \sum_{n \geq 0} a_{rn} t^n$ , for any integer  $r \geq 2$ . Then we have*

$$f^{(r)}(t) = \frac{h_0^{(r)} + \cdots + h_m^{(r)} t^m}{(1-t)^d},$$

*where  $m := \max\{s, d\}$  and*

$$h_i^{(r)} = \sum_{j=0}^s C(r-1, d, ir-j) h_j,$$

*for  $i = 0, \dots, m$ .*

We will apply this lemma to the case when  $f(t) = H_A(t)$  for a connected graded algebra  $A$  and  $f^{(r)}(t) = H_{A^{(r)}}(t)$  where  $A^{(r)}$  is the  $r$ th Veronese subring of  $A$  (i.e. the subring of elements of  $A$  with degree a multiple of  $r$ ). The following lemma is easy to see.

**Lemma 2.2.** *Suppose  $d \geq 2$ .*

- (a)  $C(a, d, 0) = 1$  for all  $a \geq 0$ .
- (b)  $C(r-1, 2, r) = r-1$ .
- (c) *If  $d \geq 3$ , then  $C(r-1, d, r) > d$ .*

**Proposition 2.3.** *Let  $A$  be a connected graded algebra with Hilbert series*

$$H_A(t) = \frac{1 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

*where  $h_i \geq 0$  for all  $i$ . Then the  $r$ th Veronese subring  $A^{(r)}$  is not cyclotomic if one of the following conditions holds.*

- (a)  $r \geq 3$  and  $H_A(t) = (1-t)^{-2}$ .
- (b)  $r$  satisfies the inequality  $C(r-1, d, r) > \max\{s, d\}$ .
- (c)  $r \geq 2$  and  $H_A(t) = (1-t)^{-d}$  and  $d \geq 3$ .

*Proof.* (a) In this case it is easy to see that  $H_{A^{(r)}}(t) = \frac{1+(r-1)t}{(1-t)^2}$ . Hence  $A^{(r)}$  is not cyclotomic when  $r \geq 3$ .

(b) Let  $m = \max\{s, d\}$ . Then  $m < C(r-1, d, r)$  by the hypothesis. By Lemma 2.1,  $h_0^{(r)} = 1$  and

$$h_1^{(r)} = C(r-1, d, r) + \sum_{j=1}^s C(r-1, d, ir-j) h_j \geq C(r-1, d, r) > m.$$

Thus the polynomial  $1 + h_1^{(r)} t + \cdots + h_m^{(r)} t^m$  has a root with absolute value strictly less than 1. Hence  $A^{(r)}$  is not cyclotomic.

(c) In this case  $s = 0$ . The assertion follows from part (a) and Lemma 2.2(c).  $\square$

Quantum polynomial rings are noetherian AS regular domains whose Hilbert series are of the form  $(1-t)^{-d}$ . The following corollary can be used to state precisely when the  $r$ th Veronese subalgebra of a quantum polynomial ring is a complete intersection.

**Corollary 2.4.** *Let  $A$  be a connected graded algebra.*

- (a) *Suppose  $A = k[t]$  where  $\deg t = 1$ . For every  $r \geq 2$ ,  $A^{(r)} \cong k[x]$  where  $\deg x = r$ . So  $A^{(r)}$  is AS regular, and consequently,  $A^{(r)}$  is cyclotomic and a classical complete intersection.*
- (b) *Suppose  $A$  is noetherian of global dimension 2 with Hilbert series  $(1-t)^{-2}$ .*
  - (i)  *$A^{(2)}$  is a classical complete intersection (and hence cyclotomic).*
  - (ii) *For every  $r \geq 3$ ,  $A^{(r)}$  is not cyclotomic. Consequently,  $A^{(r)}$  is not a complete intersection of any type.*
- (c) *Suppose the Hilbert series of  $A$  is  $(1-t)^{-d}$ . If either  $r \geq 3$  or  $d \geq 3$ , then  $A^{(r)}$  is not cyclotomic. Consequently,  $A^{(r)}$  is not a complete intersection of any type.*

*Proof.* (a) This is obvious.

(bi) This follows from the classification of quantum polynomial rings of dimension 2 and a direct computation from a nice presentation of  $A^{(2)}$  (including the generators and relations of  $A^{(2)}$ ).

(bii) This follows from Proposition 2.3(a).

(c) This follows from Proposition 2.3(b,c).  $\square$

Another special case is when  $A$  is AS regular of global dimension three and generated by two elements of degree 1.

**Lemma 2.5.** *Suppose  $H_A(t) = \frac{1}{(1-t)^2(1-t^2)}$ . If  $r \geq 3$ , then  $A^{(r)}$  is not cyclotomic. Consequently,  $A^{(r)}$  is not a complete intersection of any type.*

*Proof.* We compute the Hilbert series of  $A^{(r)} = \sum_k a_{kr} t^{kr}$ , when  $\sum_j a_j t^j$  is the expansion of  $\frac{1}{(1-t)^2(1-t^2)}$ . When  $r$  is even the Hilbert series of  $A^{(r)}$  is

$$\frac{(r^2 - 4r + 4)t^{2r} + (r^2 + 4r - 8)t^r + 4}{4(1 - t^r)^3}.$$

For the numerator to be symmetric it is necessary for  $r$  to be a positive integer with  $r^2 - 4r + 4 = \pm 4$  or 0, which happens only when  $r = 2$  or  $r = 4$ . The Hilbert series for  $r = 4$  is

$$\frac{1 + 6t^4 + t^8}{(1 - t^4)^3},$$

which is not cyclotomic. When  $r$  is odd the Hilbert series of  $A^{(r)}$  is

$$\frac{b_0 + b_1 t^r + b_2 t^{2r} + b_3 t^{3r} + b_4 t^{4r} + b_5 t^{5r} + b_6 t^{6r} + b_7 t^{7r} + b_8 t^{8r} + b_9 t^{9r}}{4(1 - t^{2r})^5}$$

where

$$\begin{aligned} b_0 &= 4, & b_1 &= 3 + 4r + r^2, & b_2 &= -16 + 8r + 4r^2, \\ b_3 &= -12 - 8r + 4r^2, & b_4 &= 24 - 24r - 4r^2, & b_5 &= 18 - 10r^2, & b_6 &= -16 + 24r - 4r^2, \\ b_7 &= -12 + 8r + 4r^2, & b_8 &= 4 - 8r + 4r^2, & b_9 &= 3 - 4r + r^2. \end{aligned}$$

For the numerator to be symmetric it is necessary that  $r$  is an odd positive integer  $\geq 3$  with  $3 - 4r + r^2 = \pm 4$  or 0, which could happen only when  $r = 3$ . The Hilbert series for  $r = 3$  is

$$\frac{1 + 6t^3 + 11t^6 - 21t^{12} - 18t^{15} + 5t^{18} + 12t^{21} + 4t^{24}}{(1 - t^6)^5},$$

which is not cyclotomic.  $\square$

**Remark 2.6.**

- (a) When  $A$  is a noetherian AS regular ring of global dimension three that is generated by two elements of degree 1, then  $H_A(t) = \frac{1}{(1-t)^2(1-t^2)}$ . We have computed many examples that indicate that  $A^{(2)}$  is a cci, and so we conjecture that  $A^{(2)}$  is always a cci in this case.
- (b) If  $A$  is noetherian and AS Gorenstein of dimension  $d$  then by [JoZ, Theorem 3.6]  $A^{(r)}$  is AS Gorenstein if and only if  $r$  divides  $\ell$  (where  $\ell$  is the AS index (see Definition 3.1 below)).

**3. INVARIANT THEORY OF AS REGULAR ALGEBRAS AND QUASI-BIREFLECTIONS**

In this section we connect the study of complete intersections with noncommutative invariant theory. First we review some definitions.

**Definition 3.1.** Let  $A$  be a connected graded algebra. We call  $A$  *Artin-Schelter Gorenstein* (or *AS Gorenstein* for short) if the following conditions hold:

- (a)  $A$  has injective dimension  $d < \infty$  on the left and on the right,
- (b)  $\text{Ext}_A^i(Ak, {}_A A) = \text{Ext}_A^i(k_A, A_A) = 0$  for all  $i \neq d$ , and
- (c)  $\text{Ext}_A^d(Ak, {}_A A) \cong \text{Ext}_A^d(k_A, A_A) \cong k(l)$  for some  $l$  (where  $l$  is called the *AS index*).

If, in addition,

- (d)  $A$  has finite global dimension, and
- (e)  $A$  has finite Gelfand-Kirillov dimension,

then  $A$  is called *Artin-Schelter regular* (or *AS regular* for short) of dimension  $d$ .

**Definition 3.2.** Let  $A$  be a noetherian algebra.

- (a) Given any  $A$ -module  $M$ , the  $j$ -number of  $M$  is defined by

$$j(M) = \min\{i \mid \text{Ext}^i(M, A) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

- (b)  $A$  is called *Auslander Gorenstein* if
  - (a)  $A$  has finite left and right injective dimension;
  - (b) for every finitely generated left  $A$ -module  $M$  and for every right  $A$ -submodule  $N \subset \text{Ext}_A^i(M, A)$ ,  $j(N) \geq i$ ,
  - (c) the above condition holds when left and right are switched.
- (c)  $A$  is called *Auslander regular* if  $A$  is Auslander Gorenstein and has finite global dimension.
- (d)  $A$  is called *Cohen-Macaulay* if, for every finitely generated (left or right)  $A$ -module  $M$ ,  $\text{GKdim}(M) + j(M) = \text{GKdim } A < \infty$ .

In a couple of places we also use dualizing complexes, as well as the Auslander and Cohen-Macaulay properties of dualizing complexes, all of which are quite technical. We refer to the paper [YZ] for these definitions. For a graded algebra, the existence of an Auslander Cohen-Macaulay dualizing complex was proved for many classes of noetherian algebras. It is conjectured that every noetherian AS regular algebra is Auslander regular.

**Lemma 3.3.** *Let  $E$  be the Ext-algebra of  $A$ . Suppose  $A$  is a nci and  $E$  has an Auslander Cohen-Macaulay dualizing complex. Then  $\text{nci}(A) \leq \text{gci}(A) < \infty$ .*

*Proof.* By using [YZ, Corollary 2.18] (for  $d_0 = 0$  and  $\text{Cdim} = \text{GKdim}$ ), we obtain  $\text{Kdim } E \leq \text{GKdim } E < \infty$ , and the assertions follow.  $\square$

Combining Theorem 1.11 with Lemma 3.3, under some reasonable conditions, we have

$$(E3.3.1) \quad cci(A) \geq gci(A) \geq nci(A).$$

Now we are ready to prove Theorem 0.6.

**Theorem 3.4.** *Assume that  $k$  is of characteristic zero. Let  $R$  be a connected graded noetherian Auslander regular algebra and  $G$  a finite subgroup of  $\text{Aut}(R)$ . If  $R^G$  is a wci, then it is cyclotomic Gorenstein.*

*Proof.* By [JoZ, Lemma 5.5], the Hilbert series of  $R^G$  is a rational function. By Theorem 1.11(c),  $R^G$  is cyclotomic. It remains to show that  $R^G$  is AS Gorenstein.

Write  $H_{R^G}(t) = p(t)/q(t)$ . By the proof of Theorem 1.11(c), every root of  $p(t)$  is a root of unity. Since  $p(t)$  is an integral polynomial, we have  $p(t^{-1}) = \pm t^{d_1} p(t)$  for some  $d_1$ . Since  $R^G$  has finite GK-dimension, every root of  $q(t)$  is a root of unity (see the proof of [StZ, Corollary 2.2]). Hence  $q(t^{-1}) = \pm t^{d_2} q(t)$  for some  $d_2$ . It follows from [JoZ, Theorem 6.4] that  $R^G$  is AS Gorenstein.  $\square$

When  $A = R^G$  where  $R$  is a connected graded noetherian Auslander regular algebra and  $G$  is a finite subgroup of  $\text{Aut}(R)$ , we can modify the diagram (E1.11.2) a little by changing “cyclotomic” to “cyclotomic Gorenstein”.

This is a good place to mention a natural question.

**Question 3.5.** Let  $A$  be a noetherian connected graded algebra that is either a nci or a gci. Under what conditions must  $A$  be AS Gorenstein (or Gorenstein)?

Example 6.3 shows that  $A$  can be both a nci and a gci yet still not be Gorenstein. However Theorem 3.4 says that for  $A = R^G$ , where  $R$  is a noetherian Auslander regular algebra, then  $A$  must be AS Gorenstein.

**Definition 3.6.** Let  $A$  be a noetherian connected graded (AS regular) algebra of GK-dimension  $n$ . Let  $g \in \text{Aut}(A)$ .

- (a) [KKZ1, Definition 2.2] We call  $g$  a *quasi-reflection* if its trace has the form:

$$\text{Tr}_A(g, t) = \frac{1}{(1-t)^{n-1}q(t)}$$

where  $q(t)$  is an integral polynomial with  $q(1) \neq 0$ .

- (b) We call  $g$  a *quasi-bireflection* if its trace has the form:

$$\text{Tr}_A(g, t) = \frac{1}{(1-t)^{n-2}q(t)}$$

where  $q(t)$  is an integral polynomial with  $q(1) \neq 0$ .

As the classical case, a quasi-reflection is also viewed as a quasi-bireflection for convenience.

**Definition 3.7.** Let  $A$  be a noetherian AS regular algebra and let  $W$  be a subgroup of  $\text{Aut}(A)$ .

- (a) The *cci- $W$ -bound* of  $A$  is defined to be

$$cci(W/A) = \sup\{cci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } cci(A^G) < \infty\}.$$

- (b) The *gci- $W$ -bound* of  $A$  is defined to be

$$gci(W/A) = \sup\{gci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } gci(A^G) < \infty\}.$$

(c) The *nci*-*W*-bound of  $A$  is defined to be

$$nci(W/A) = \sup\{nci(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } nci(A^G) < \infty\}.$$

(d) The *cyc*-*W*-bound of  $A$  is defined to be

$$cyc(W/A) = \sup\{cyc(A^G) \mid \forall \text{ finite subgroups } G \subset W \text{ such that } cyc(A^G) < \infty\}.$$

**Example 3.8.** Let  $A$  be a noetherian AS regular algebra of global dimension two that is generated in degree 1. Then by [KKZ4, Theorem 0.1],

$$cci(\text{Aut}(A)/A) = gci(\text{Aut}(A)/A) = nci(\text{Aut}(A)/A) = cyc(\text{Aut}(A)/A) = 1.$$

#### 4. GRADED TWISTS

In this section we will prove Theorem 0.3. Let  $A$  be an  $\mathbb{Z}^d$ -graded algebra and  $\text{Aut}_{\mathbb{Z}^d}(A)$  be the group of  $\mathbb{Z}^d$ -graded algebra automorphisms of  $A$ . Let  $\{\tau_1, \dots, \tau_d\}$  be a set of commuting elements in  $\text{Aut}_{\mathbb{Z}^d}(A)$ . We define a twisting system of  $A$  by  $\tau_g = \tau_1^{g_1} \cdots \tau_d^{g_d}$  for every  $g = (g_1, \dots, g_d) \in \mathbb{Z}^d$ . Then  $\tau = \{\tau_g \mid g \in \mathbb{Z}^d\}$  is a left (respectively, right) twisting system in the sense of [Zh2, Definition 2.1]. Let  $A^\tau$  be the graded twisted algebra by the twisting system  $\tau$  [Zh2, Proposition and Definition 2.3]. Let  $A - \text{Gr}$  be the category of graded left  $A$ -modules. There is also a definition of twisted graded module  $M^\tau$  [Zh2, Proposition and Definition 2.6]. In this paper we are working with left modules instead of right modules. Then the assignment  $F : M \mapsto M^\tau$  defines an equivalence of categories [Zh2, Theorem 3.1]

$$A - \text{Gr} \cong A^\tau - \text{Gr}.$$

Let  $(g)$  denote the degree  $g$  shift of a graded module. Let  $M$  and  $N$  be finitely generated graded left modules over a left noetherian ring  $A$ . Then we have

$$\text{Ext}_A^i(M, N) \cong \bigoplus_{g \in \mathbb{Z}^d} \text{Ext}_{A - \text{Gr}}^i(M, N(g)).$$

**Lemma 4.1.** *Let  $A$  be an  $\mathbb{N}^d$ -graded finitely generated algebra such that  $A$  becomes a connected  $\mathbb{N}$ -graded when taking the total degree. Suppose  $\tau = \{\tau_1, \dots, \tau_d\}$  is a set of commuting elements in  $\text{Aut}_{\mathbb{Z}^d}(A)$ . Then  $A$  is a gci if and only if  $A^\tau$  is.*

*Proof.* Let  $B = A^\tau$ . It suffices to show that  $\text{Ext}_A^i(k, k) \cong \text{Ext}_B^i(k, k)$  as graded vector space. Let  $F$  be the functor sending  $M$  to  $M^\tau$ . Since  $A$  is connected graded by using the total degree, one can check that the only simple graded module over  $A$  is  $k(g)$  for  $g \in \mathbb{Z}^d$ . Using this fact we obtain that  $F({}_A k(g)) \cong_B k(g)$  for all  $g \in \mathbb{Z}^d$ . Hence

$$\begin{aligned} \text{Ext}_A^i(k, k) &= \bigoplus_{g \in \mathbb{Z}^d} \text{Ext}_{A - \text{Gr}}^i(k, k(g)) \cong \bigoplus_{g \in \mathbb{Z}^d} \text{Ext}_{B - \text{Gr}}^i(F(k), F(k(g))) \\ &\cong \bigoplus_{g \in \mathbb{Z}^d} \text{Ext}_{B - \text{Gr}}^i(k, k(g)) = \text{Ext}_B^i(k, k). \end{aligned}$$

□

**Lemma 4.2.** *Let  $A$  and  $\tau$  be as in Lemma 4.1. Let  $G$  be a subgroup of  $\text{Aut}_{\mathbb{Z}^d}(A)$  such that every element  $g \in G$  commutes with  $\tau_i$  for all  $i = 1, \dots, d$ . Then*

- (a)  $G$  is a subgroup of  $\text{Aut}_{\mathbb{Z}^d}(A^\tau)$  under the identification  $A = A^\tau$  (as a graded vector space).

- (b) The restriction of  $\tau$  on  $A^G$  defines a twisting system of  $A^G$ , which we still denote by  $\tau$ .
- (c)  $(A^\tau)^G$  is a graded twist of  $A^G$  by  $\tau$ .

*Proof.* Let  $B = A^\tau$  with multiplication  $*$ . By definition,  $A = B$  as graded vector spaces. The difference between  $A$  and  $B$  is in their multiplication, see [Zh2, Proposition and Definition 2.3].

(a) For every  $g \in G$ , define  $g_B : B \rightarrow B$  to be the graded vector space automorphism  $g$ . We need to show that this is an algebra homomorphism. For any  $x, y \in B$ ,

$$\begin{aligned} g_B(x * y) &= g(\tau^{\deg y}(x)y) = g(\tau^{\deg y}(x))g(y) \\ &= \tau^{\deg y}(g(x))g(y) = g(x) * g(y) = g_B(x) * g_B(y). \end{aligned}$$

Therefore  $g_B$  is a graded algebra automorphism of  $B$ , and the assertion follows.

(b) Since  $\tau_i$  commutes with  $G$ , each  $\tau_i$  induces an automorphism of  $A^G$  by restriction. Since the restrictions of  $\tau_i$  on  $A^G$  commute with each other,  $\tau$  defines a twisting system of  $A^G$ .

(c) As graded subspaces of  $A$ , we have  $(A^G)^\tau = A^G = (A^\tau)^G$ . By using the twisting, we see that the multiplications of  $(A^G)^\tau$  and  $(A^\tau)^G$  are the same. In fact,  $(A^G)^\tau = (A^\tau)^G$  as subalgebras of  $A^\tau$ .  $\square$

**Proposition 4.3.** *Let  $(A, \tau, G)$  be as in Lemma 4.2.*

- (a)  $(A^\tau)^G$  is a gci if and only if  $A^G$  is.
- (b) For every  $g \in G$ ,  $\text{Tr}_A(g_A, t) = \text{Tr}_{A^\tau}(g_{A^\tau}, t)$ .
- (c)  $G \subset \text{Aut}_{\mathbb{Z}^d}(A) \subset \text{Aut}(A)$  is generated by quasi-bireflections of  $A$  if and only if  $G \subset \text{Aut}_{\mathbb{Z}^d}(A^\tau) \subset \text{Aut}(A^\tau)$  is generated by quasi-bireflections of  $A^\tau$ .

*Proof.* (a) By Lemma 4.2(c),  $(A^\tau)^G$  is a graded twist of  $A^G$ . The assertion follows from Lemma 4.1.

(b) This follows from the fact that  $A = A^\tau$  as an  $\mathbb{N}$ -graded vector space and  $g_A = g_{A^\tau}$  as a graded vector space automorphism.

(c) Follows from part (b) and the definition of quasi-bireflection.  $\square$

Now we consider the skew polynomial ring  $B = k_{p_{ij}}[x_1, \dots, x_d]$  which is generated by  $\{x_1, \dots, x_d\}$  and subject to the relations

$$x_j x_i = p_{ij} x_i x_j$$

for all  $i < j$  where  $\{p_{ij}\}_{1 \leq i < j \leq d}$  is a set of nonzero scalars.

This is a  $\mathbb{Z}^d$ -graded algebra when setting  $\deg x_i = (0, \dots, 0, 1, 0, \dots, 0) =: e_i$ , where 1 is in the  $i$ th position. We say an automorphism  $g$  of  $A$  is *diagonal* if  $g(x_i) = a_i x_i$  for some  $a_i \in k^\times$ . Then every  $\mathbb{Z}^d$ -graded algebra automorphism of  $A$  is diagonal. As a consequence,  $\text{Aut}_{\mathbb{Z}^d}(A) = (k^\times)^d$ .

**Lemma 4.4.** *Let  $B$  be the skew polynomial ring  $k_{p_{ij}}[x_1, \dots, x_d]$  and  $G$  a finite subgroup of  $\text{Aut}_{\mathbb{Z}^d}(A)$ .*

- (a) [KKZ2, Lemma 3.2]  $B$  is a  $\mathbb{Z}^d$ -graded twist of the commutative polynomial ring by  $\tau = (\tau_1, \dots, \tau_d)$  where  $\tau_i$  is defined by  $\tau_i(x_s) = \begin{cases} p_{is} x_s & i < s \\ x_s & i \geq s \end{cases}$  for all  $s$ .
- (b)  $G$  commutes with  $\tau_i$ .



(c) If  $B^G$  is a gci, then  $G$  is generated by quasi-bireflections.

*Proof.* (a,b) are checked directly.

(c) By part (a)  $B = A^\tau$  where  $A = k[x_1, \dots, x_d]$ . If  $B^G$  is a gci, by Proposition 4.3(a),  $A^G$  is a gci. Since  $A = k[x_1, \dots, x_d]$ , by the Kac-Watanabe-Gordecv Theorem ([KW] or [G1]),  $G \subset \text{Aut}(A)$  is generated by bireflections (which are also quasi-bireflections in the sense of Definition 3.6(b)). By Proposition 4.3(c),  $G \subset \text{Aut}(B)$  is generated by quasi-bireflections of  $B$ .  $\square$

The following lemma is well-known, see for example [AC].

**Lemma 4.5.** *Let  $B$  be the skew polynomial ring  $k_{p_{ij}}[x_1, \dots, x_d]$ . Suppose that  $p_{ij}p_{st} \neq 1$  for any  $i < j$  and  $s < t$ . Then  $\text{Aut}(B) = (k^\times)^d$ .*

*Proof.* First of all  $(k^\times)^d = \text{Aut}_{\mathbb{Z}^d}(B) \subset \text{Aut}(B)$ . It remains to prove that every  $\mathbb{Z}$ -graded algebra automorphism  $g$  of  $B$  is diagonal. Since  $p_{ij} \neq 1$  for all  $i < j$ , by [KKZ2, Lemma 3.5(e)] there is a permutation  $\sigma \in S_d$  and  $c_s \in k^\times$  for each  $s$  such that

$$g(x_i) = c_i x_{\sigma(i)} \quad \text{for every } i = 1, \dots, d.$$

We need to show that  $\sigma$  is the identity. Suppose  $\sigma$  is not the identity. There are distinct integers  $i_1, \dots, i_k$  in the set  $\{1, \dots, d\}$  for some  $k > 1$  such that

$$\sigma : i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_k \rightarrow i_1.$$

If  $k = 2m$  is even, then  $p_{i_1 i_{m+1}} = p_{\sigma^m(i_1) \sigma^m(i_{m+1})} = p_{i_{m+1} i_1}$ . Hence  $p_{i_1 i_{m+1}}^2 = 1$ , a contradiction. If  $k$  is odd, then  $k \geq 3$ . Hence

$$p_{i_1 i_2} = p_{i_2 i_3} = p_{i_3 i_4}$$

when  $k > 3$ , or

$$p_{i_1 i_2} = p_{i_2 i_3} = p_{i_3 i_1}$$

when  $k = 3$ . In both cases, one can find  $i < j$  and  $s < t$  such that  $p_{ij}p_{st} = 1$ , a contradiction. Therefore  $\sigma$  is the identity, completing the proof.  $\square$

Now we are ready to prove Theorem 0.3.

*Proof of Theorem 0.3.* Let  $C$  denote the algebra  $k_q[x_1, \dots, x_d]$ . Then  $C$  is a special case of  $k_{p_{ij}}[x_1, \dots, x_d]$  by taking  $p_{ij} = q$  for all  $i < j$ . Since  $q^2 \neq 1$  (or  $q \neq \pm 1$ ), Lemma 4.5 says that  $\text{Aut}(C) = (k^\times)^d$ . Let  $G$  be a finite subgroup of  $\text{Aut}(C)$ . Then  $G \subset (k^\times)^d$ , and the assertion follows from Lemma 4.4(c).  $\square$

## 5. SOME FIXED SUBRINGS OF THE SKLYANIN ALGEBRA

Let  $A = A(a, b, c)$  be the 3-dimensional Sklyanin algebra, i.e. the algebra generated by  $x, y, z$  with relations

$$ax^2 + byz + cxy = 0$$

$$ay^2 + bzx + cxz = 0$$

$$az^2 + bxy + cyx = 0.$$

By [ATV] the algebra  $A$  is noetherian and AS regular except when  $a^3 = b^3 = c^3$  or when two of the three parameters  $a, b, c$  are zero. We assume throughout that parameters  $a, b, c$  are generic enough so that none of  $a, b, c$  is zero and the cubes  $a^3, b^3, c^3$  are not all equal, and we assume further that  $A$  is not PI; with these

assumptions  $A$  is AS regular. Then by [Sm, Example 10.1], its Koszul dual  $A^!$  is generated by  $X, Y, Z$  with relations:

$$\begin{aligned} cYZ - bZY &= 0 & bX^2 - aYZ &= 0 \\ cZX - bXZ &= 0 & bY^2 - aZX &= 0 \\ cXY - bYX &= 0 & bZ^2 - aXY &= 0. \end{aligned}$$

By Koszul duality, groups  $\text{Aut}(A)$  and  $\text{Aut}(A^!)$  are anti-isomorphic. We can determine  $\text{Aut}(A)$  by computing  $\text{Aut}(A^!)$ .

Let the matrix  $(\alpha_{ij})$  represent the linear map  $\sigma$  that takes

$$\begin{aligned} \sigma(X) &= \alpha_{11}X + \alpha_{21}Y + \alpha_{31}Z, \\ \sigma(Y) &= \alpha_{12}X + \alpha_{22}Y + \alpha_{32}Z, \\ \sigma(Z) &= \alpha_{13}X + \alpha_{23}Y + \alpha_{33}Z. \end{aligned}$$

**Lemma 5.1.** *The automorphisms of  $A^!$  are of the following forms:*

$$g_1 = \begin{pmatrix} \alpha\omega & 0 & 0 \\ 0 & \alpha\omega^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, g_2 = \begin{pmatrix} 0 & \alpha\omega & 0 \\ 0 & 0 & \alpha\omega^2 \\ \alpha & 0 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 0 & \alpha\omega \\ \alpha\omega^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix},$$

where  $\alpha$  is arbitrary and  $\omega$  satisfies  $\omega^3 = 1$ . The graded algebra automorphisms of  $A$  are transposes of these matrices, so of the same forms.

*Proof.* Recall that we assume  $a, b, c$  are generic, and since  $A$  is not PI we may assume  $b \neq c$ . Since  $b \neq 0$  without loss of generality we may assume that  $b = 1$  and  $c \neq 1$ . Then since  $\sigma$  must preserve the equations:  $YZ - bZY = 0$ ,  $ZX - bXZ = 0$  and  $XY - bYX = 0$  in  $A^!$  we see that the entries  $\alpha_{i,j}$  of  $\sigma$  must satisfy the equations:

$$\begin{aligned} (1+c)\alpha_{1,2}\alpha_{2,1} + a\alpha_{3,1}\alpha_{3,2} &= 0 \\ (1+c)\alpha_{2,2}\alpha_{3,1} + a\alpha_{1,1}\alpha_{1,2} &= 0 \\ (1+c)\alpha_{3,2}\alpha_{1,1} + a\alpha_{2,1}\alpha_{2,2} &= 0 \\ \\ (1+c)\alpha_{2,3}\alpha_{1,1} + a\alpha_{3,1}\alpha_{3,3} &= 0 \\ (1+c)\alpha_{1,3}\alpha_{3,1} + a\alpha_{2,1}\alpha_{2,3} &= 0 \\ (1+c)\alpha_{3,3}\alpha_{2,1} + a\alpha_{1,1}\alpha_{1,3} &= 0 \\ \\ (1+c)\alpha_{1,2}\alpha_{3,3} + a\alpha_{2,3}\alpha_{2,2} &= 0 \\ (1+c)\alpha_{2,2}\alpha_{1,3} + a\alpha_{3,3}\alpha_{3,2} &= 0 \\ (1+c)\alpha_{3,2}\alpha_{2,3} + a\alpha_{1,3}\alpha_{1,2} &= 0. \end{aligned}$$

Using Maple it is easy to see that the only nonsingular matrices satisfying these equations are matrices of the forms:

$$g_1 = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{pmatrix}, g_2 = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & \beta \\ \alpha & 0 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 0 & \gamma \\ \beta & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}.$$

From the relations  $X^2 - aYZ = 0$ ,  $Y^2 - aZX = 0$  and  $Z^2 - aXY = 0$  we see that  $\alpha^3 = \beta^3 = \gamma^3$ . Taking  $\alpha \neq 0$  arbitrary, these final three relations show that  $\sigma$  must be of one of the three forms indicated.  $\square$

If  $a, b, c$  are not generic, there are more graded algebra automorphisms of  $A$ . To specify  $\omega$  and  $\alpha$ , we also use  $g_i(\alpha, \omega)$  for the matrices (or the automorphisms) listed in Lemma 5.1.

By [KKZ1, Corollary 6.3],  $A$  has no quasi-reflection of finite order. By [KKZ2, Corollary 4.11], if  $G$  is a finite subgroup of  $\text{Aut}(A)$  and  $A^G$  is Gorenstein, then  $\text{hdet } g = 1$  for all  $g \in G$ . For the homological determinant of  $g \in G$  to be 1, the trace of the induced map, denoted by  $g^!$  on  $A^!$  must have the coefficient of  $t^3$  equal to 1 by [JoZ, Lemma 2.6]. The following lemma follows from a direct computation.

**Lemma 5.2.** *Let  $A = A(a, b, c)$  and  $SL(A)$  be the subgroup of  $\text{Aut}(A)$  generated by  $g_1, g_2$  and  $g_3$  with  $\alpha^3 = \omega^3 = 1$ .*

- (a) *The order of  $SL(A)$  is 27.*
- (b)  *$\text{hdet } g = 1$  for all  $g \in SL(A)$ .*
- (c) *If  $G$  is a subgroup of  $\text{Aut}(A)$  with trivial homological determinant, then  $G$  is a subgroup of  $SL(A)$ .*

Now we consider only  $g \in G$  with trivial homological determinant. For  $g$  to be a quasi-bireflection we need  $t = 1$  to be a root of  $\text{Tr}_A(g, t)$  of multiplicity 1, which occurs when  $t = -1$  is a root of  $\text{Tr}_{A^!}(g^!, t)$  of multiplicity 1, and this happens in each case when  $\alpha^3 = 1$  and in the first case when, in addition, we have  $\omega \neq 1$  (i.e. we eliminate the case of a scalar matrix); then, in each case, the automorphism  $g$  is a quasi-bireflection with  $\text{Tr}_A(g, t) = 1/(1 - t^3)$  and  $\text{hdet } g = 1$ .

**Lemma 5.3.** *Retaining the notation above we have:*

- (a)  *$g$  is a quasi-bireflection of  $A$  if and only if  $g$  is of the form  $g_1, g_2$  or  $g_3$  with  $\alpha^3 = \omega^3 = 1$ , and  $\omega \neq 1$  when  $g = g_1$ . As a consequence,  $g \in SL(A)$ .*
- (b) *If  $g$  is a quasi-bireflection of  $A$ , then  $\text{Tr}_A(g, t) = (1 - t^3)^{-1}$ .*

We can classify all subgroups of  $SL(A)$ .

**Lemma 5.4.** *The complete list of subgroups of  $SL(A)$  is as follows:*

- (a)  $\{1\}$ .
- (b) *order 3 subgroups generated by single element, namely,*

$$\langle g_1(\alpha, \omega) \rangle, \quad \langle g_2(\alpha, \omega) \rangle, \quad \text{and} \quad \langle g_3(\alpha, \omega) \rangle$$
*for any pair  $(\alpha, \omega)$  with  $\alpha^3 = \omega^3 = 1$ . In the case of  $g_1$ ,  $(\alpha, \omega) \neq (1, 1)$ .*
- (c) *order 9 subgroup  $G_1 := \langle g_1(\alpha, \omega) \mid \alpha^3 = \omega^3 = 1 \rangle$ .*
- (d) *order 9 subgroups*

$$G_2 := \langle g_1(\alpha \neq 1, \omega = 1), g_2 \rangle, \quad \text{and} \quad G_3 := \langle g_1(\alpha \neq 1, \omega = 1), g_3 \rangle.$$
- (e) *the whole group  $SL(A)$ .*

*Proof.* If the order of the subgroup is  $\leq 3$ , then clearly we get cases (a) and (b). It is obvious that the subgroups  $G_1, G_2, G_3$  in parts (c,d) are of order 9. Now assume that  $G$  is a subgroup of  $SL(A)$  of order 9, which is not of the form in part (c). So  $G$  contains either  $g_2$  or  $g_3$ . By symmetry, we assume that  $g_2 \in G$ . Since the order of every element in  $G$  is either 1 or 3,  $G \cong C_3 \times C_3$ . So  $G$  is abelian, so there are two elements of order  $g_2$ , say  $a = g_2(\alpha_1, \omega_1)$  and  $b = g_2(\alpha_2, \omega_2)$ . Then  $ab^{-1} = g_1(\alpha, \omega)$ . Since the order of  $G$  is 9,  $\omega = 1$ . Thus we have the group  $G_2$  or  $G_3$ .  $\square$

**Theorem 5.5.** *Retaining the notation above, we have  $\text{cyc}(\text{Aut}(A)/A) = 1$ , and the following are equivalent for subgroups  $\{1\} \neq G \subset SL(A)$ .*

- (a)  $G$  is not  $\langle g_1(\alpha, 1) \rangle$ .
- (b)  $G$  is generated by quasi-bireflections.
- (c)  $A^G$  is cyclotomic Gorenstein.

*Proof.* Since  $\text{Tr}_A(g_1(\alpha, 1), t) = (1 - \alpha t)^3$ ,  $g_1(\alpha, 1)$  is not a quasi-bireflection when  $\alpha \neq 1$ . Hence  $G = \langle g_1(\alpha, 1) \rangle$  is not generated by quasi-bireflections. The fixed subring  $A^G$  is the Veronese  $A^{(3)}$ . By Corollary 2.4(c),  $A^G$  is not cyclotomic.

It is straightforward to check that all other groups  $G$  are generated by quasi-bireflections. So it remains to show that  $A^G$  is cyclotomic Gorenstein in each of these cases.

If  $G$  has order 3 with generator  $g$  where  $g$  is either  $g_2, g_3$  or  $g_1$  with  $\omega \neq 1$ , then the fixed ring  $A^g$  has Hilbert series

$$\frac{1}{3(1-t)^3} + \frac{2}{3(1-t^3)} = \frac{1-t+t^2}{(1-t)^2(1-t^3)} = \frac{1-t^6}{(1-t)(1-t^2)(1-t^3)^2}.$$

So  $A^G$  is cyclotomic Gorenstein. If  $G$  is the any group of order 9 in Lemma 5.4, then the Hilbert series of  $A^G$  is

$$\begin{aligned} H_{A^G}(t) &= \frac{1}{9} \left( \frac{1}{(1-t)^3} + \frac{1}{(1-\omega t)^3} + \frac{1}{(1-\omega^2 t)^3} + \frac{6}{(1-t^3)} \right) \\ &= \frac{1+t^3+t^6}{(1-t^3)^3} = \frac{1-t^9}{(1-t^3)^4}. \end{aligned}$$

where  $\omega$  is a primitive 3rd root of unity. So  $A^G$  is cyclotomic Gorenstein.

Finally, if  $G = SL(A)$ , then the Hilbert series of  $A^G$  is

$$\begin{aligned} H_{A^G}(t) &= \frac{1}{27} \left( \frac{1}{(1-t)^3} + \frac{1}{(1-\omega t)^3} + \frac{1}{(1-\omega^2 t)^3} + \frac{24}{(1-t^3)} \right) \\ &= \frac{1-t^3+t^6}{(1-t^3)^3} = \frac{1+t^9}{(1-t^3)^2(1-t^6)} = \frac{1-t^{18}}{(1-t^3)^2(1-t^6)(1-t^9)}. \end{aligned}$$

So  $A^G$  is cyclotomic Gorenstein.

Finally, by the above Hilbert series, we see that  $\text{cyc}(\text{Aut}(A)/A) = 1$ . □

We make the following conjecture.

**Conjecture 5.6.** *Let  $A = A(a, b, c)$  where  $a, b, c$  are generic. Then the following are equivalent for any finite subgroup  $G \subset \text{Aut}(A)$ .*

- (a)  $A^G$  is a cci.
- (b)  $A^G$  is a gci.
- (c)  $A^G$  is a nci.
- (d)  $A^G$  is cyclotomic Gorenstein.
- (e)  $G$  is generated by quasi-bireflections.

## 6. EXAMPLES AND QUESTIONS

In this section we collect some examples and questions which indicate the difference between the commutative and the noncommutative situations. First we show that cci, gci and nci are different.

**Lemma 6.1.** *Suppose  $A$  is a finite dimensional algebra. If  $A \cong C/(\Omega_1, \dots, \Omega_n)$  where  $C$  has finite global dimension and  $\{\Omega_1, \dots, \Omega_n\}$  is a sequence of regular normal elements in  $C$ , then  $C$  is noetherian, AS regular, Auslander regular and Cohen-Macaulay, and  $A$  is Frobenius.*

*Proof.* Since  $A$  is noetherian,  $C$  is noetherian by [Le, Proposition 3.5 (a)]. Furthermore  $C$  has enough normal elements in the sense of [Zh1, p. 392]. By [Zh1, Theorem 0.2],  $C$  is AS regular, Auslander regular and Cohen-Macaulay. By Rees's lemma [Le, Proposition 3.4(b)],  $A$  is Gorenstein of injective dimension 0. Hence  $A$  is Frobenius.  $\square$

**Example 6.2.** Let  $A = k\langle x, y \rangle / (x^2, xy, y^2)$ . This is a finite dimensional Koszul algebra with Hilbert series  $1+2t+t^2$ , and  $A$  is finitely generated and noetherian. The Ext-algebra  $E = E(A)$  is isomorphic to  $k\langle x, y \rangle / (yx)$  with Hilbert series  $(1-t)^{-2}$ . By definition,  $gci(A) = \text{GKdim } E = 2$  and  $A$  is a gci. It is well-known that  $E$  is not (left or right) noetherian,  $A$  is not a nci (and  $\text{Kdim } E = \infty$ ). So  $nci(A) = \infty$ . Since  $A$  is finite dimensional and not Frobenius,  $A$  is not a cci [Lemma 6.1] and  $cci(A) = \infty$ . Since  $H_A(t) = \frac{(1-t^2)^2}{(1-t)^2}$ ,  $cyc(A) = 2$ .

**Example 6.3.** Let  $R$  be the connected graded Koszul noetherian algebra of global dimension four that is not AS regular given in [RS, Theorem 1.1]. Its Hilbert series is  $H_R(t) = (1-t)^{-4}$ . Let  $A$  be the Koszul dual of  $R$ . Since  $R$  is not AS regular,  $A$  is not Frobenius [Sm, Theorem 4.3 and Proposition 5.10]. Hence  $A$  is not a cci by Lemma 6.1. However, the Ext-algebra of  $A$  is  $R$ , which is noetherian and has GK-dimension 4. Consequently,  $A$  is both a gci and nci. It is easy to see that  $gci(A) = cyc(A) = 4$ .

An example of Stanley shows that, even in the commutative case,  $cci$  and “cyclotomic Gorenstein” are different.

**Example 6.4.** [St2, Example 3.9] Let  $A$  be the connected commutative algebra

$$k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]/I$$

with  $\deg(x_i) = 1$  where

$$I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5, x_1^2x_4 - x_5x_6x_7, x_1^3 - x_3x_5x_7).$$

This is a normal Gorenstein domain, but not a complete intersection (so  $gci(A) = \infty$ ). Its Hilbert series is

$$H_A(t) = \frac{(1+t)^3}{(1-t)^4} = \frac{(1-t^2)^3}{(1-t)^7}.$$

Hence it is cyclotomic Gorenstein and  $cyc(A) = 3$ .

Many examples and Theorem 1.11 indicate that being a cci could be the strongest among all different versions of a noncommutative complete intersection. In this direction, we need to answer the following question.

**Question 6.5.** Suppose  $\text{char } k = 0$ . If  $A$  is a cci, then is  $A$  a nci?

By Theorem 1.10(c), this conjecture is true when  $cci(A) \leq 1$ , and is unsolved for  $cci(A) \geq 2$ . If Question 6.5 has a positive answer, then we have the following diagram.

(E6.5.1)

$$\begin{array}{ccc}
cci & \xrightarrow[\text{Conjecture 6.5}]{\text{Example 6.3}} & nci \\
\downarrow \text{Theorem 1.11(a)} \quad \text{Example 6.3} & & \downarrow \text{Theorem 1.11(b)} \\
gci & \xrightarrow{\text{Theorem 1.11(b)}} & wci \\
& & \downarrow \text{Theorem 3.4} \\
& & \text{cyclotomic Gorenstein}
\end{array}$$

The next two examples concern the noncommutative version of a bireflection. We would like to show that quasi-bireflection is a good generalization in the noncommutative setting in order to answer Question 0.2.

**Example 6.6.** Let  $B = k_{-1}[x, y, z]$  where all the variables  $(-1)$ -skew commute. Let  $V = B_1 = kx + ky + kz$ . Consider

$$g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which has eigenvalues  $-1, i, -i$ . Hence  $g|_V$  is not a classical bireflection of the vector space  $V$ . By an easy computation,  $\text{hdet } g = 1$  and

$$\text{Tr}_A(g, t) = 1/((1+t)(1-t^2)) = \text{Tr}_A(g^2, t) = \text{Tr}_A(g^3, t),$$

as a consequence,  $g$  is a quasi-bireflection. A computation shows that  $A^g$  is isomorphic to

$$k[X, Y, Z, W]/(W^2 - (X^2 + 4Y^2)Z),$$

which is a commutative complete intersection. To match our terminology for quasi-reflections in [KKZ1], we might call  $g$  a mystic quasi-bireflection because  $g$  is a quasi-bireflection but not a classical bireflection of  $V$ .

**Example 6.7.** Let  $B = k_{-1}[x, y, z, w]$  where all the variables  $(-1)$ -skew commute. Let  $V = B_1 = kx + ky + kz + kw$ . Let

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then  $g$  is a classical bireflection of the vector space  $V$  but  $g$  is not a quasi-bireflection, since its trace is  $\text{Tr}(g, t) = 1/(1+t^2)^2$ . The fixed subring is Gorenstein because  $\text{hdet } g = 1$ . The Hilbert series of the fixed ring is

$$H_{B^g}(t) = \frac{1 - 2t + 4t^2 - 2t^3 + t^4}{(1-t)^4(1+t^2)^2}$$

so that  $B^g$  is Gorenstein, but not cyclotomic Gorenstein. Consequently,  $B^G$  is not a noncommutative complete intersection of any type.

In [WR] all groups  $G$  so that  $\mathbb{C}[x, y, z]^G$  is a complete intersection are completely determined. In the noncommutative case, we can ask:

**Question 6.8.** Let  $A$  be a noetherian AS regular algebra of global dimension three that is generated in degree 1. Determine all finite subgroups  $G \subset \text{Aut}(A)$  such that  $A^G$  is a noncommutative complete intersection (of certain a type).

This question was answered for global dimension 2 in [KKZ4, Theorem 0.1]. The problem of determining which groups  $G$  of graded automorphisms of  $A = k[x_1, \dots, x_n]$  have the property that  $A^G$  is a hypersurface was solved in [N1, N2] (see also [NW, Theorem 7]). So we can ask

**Question 6.9.** Let  $A$  be a noetherian AS regular algebra of global dimension at most three that is generated in degree 1. Determine all finite subgroups  $G \subset \text{Aut}(A)$  such that  $A^G$  is a hypersurface (of certain type).

Another closely related question is

**Question 6.10.** If  $\text{nci}(A) = 1$ , then is  $A$  cyclotomic Gorenstein?

The complete intersections  $A^G$  for  $A = \mathbb{C}[x, y, z]$  and  $G$  an abelian subgroup of  $GL(n, \mathbb{C})$  are computed in [W3]. We can ask, in the noncommutative case, what can we say when  $G$  is abelian.

In [WR] complete intersections  $\mathbb{C}[x, y, z]^G$  are considered; here  $\mathbb{C}[x, y, z]^G$  is a complete intersection if and only if the minimal number of algebra generators of  $\mathbb{C}[x, y, z]^G$  is  $\leq 5$  [NW, Corollary p. 107]. A complete intersection  $A^G$  for  $A = k[x_1, \dots, x_n]$  always has  $\leq 2n - 1$  generators (but for  $n=4$  there is an example of  $A^G$  not a complete intersection but the number of generators is 7). Are there noncommutative version(s) of these results? What nice properties distinguish noncommutative complete intersections from other AS Gorenstein rings?

**Remark 6.11.** To do noncommutative algebraic geometry and/or representation theory (e.g. support varieties and so on) related to noncommutative complete intersection rings sometimes it is convenient to assume that the Ext-algebra of  $A$  has nice properties. For example, one might want to assume that  $A$  is a nci and  $E(A)$  has an Auslander Cohen-Macaulay rigid dualizing complex. Together with these extra hypotheses, this should be a good definition of noncommutative complete intersection. So we conclude with the following question.

**Question 6.12.** Let  $A$  be a connected graded noetherian algebra that is a nci. Suppose that the Ext-algebra  $E(A)$  has an Auslander Cohen-Macaulay rigid dualizing complex. Then is  $A$  Gorenstein?

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